WEIGHTED STRONG PRODUCT GRAPHS BASED ON FREQUENCY EQUILIBRIUM

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Abstract:

With the emergence of large-scale and time-varying signals on graphs, product graphs have recently been used to handle these signals and approximate data topologies by integrating spatial and temporal factor graphs. Developing effective frequency analysis methods is then a fundamental issue in signal processing on product graphs. However, given the specific structure of product graphs, we theoretically show that the frequency gathering problem is highly likely to occur, potentially undermining the efficiency of frequency analysis. To address this issue, this paper proposes a novel weighted strong product graph, utilizing a weighted combination of Cartesian and Kronecker product graphs. We introduce the concept of frequency equilibrium to quantify the extent of frequency gathering in product graphs. We then derive the frequency characteristics of weighted strong product graphs and design a weighting minimization model to determine the optimal weights. Experimental results demonstrate that the proposed weighted strong product graphs and weighting model achieve superior frequency equilibrium compared to other product graphs.

Keywords:

product graphs, frequency gathering problem, frequency equilibrium, weighted strong product graphs

1. Introduction

Graphs provide general representations for various types of real-world data, including their geometric and relational structures. A variety of methods has emerged to analyze what is known as graph signals, leading to the development of the emerging field of graph signal processing [1, 2]. Fundamental graph signal processing encompasses frequency analysis, graph learning, and graph filtering. With the advent of the big data era, as the number of nodes in a graph increases, graph signal processing operations demand more memory and computational resources. One solution to this challenge is the application of graph operations in neural networks, where graph neural networks have become prevalent deep learning models for analyzing structured data [3, 4]. Another recently discussed solution is the use of product graphs, which leverage the product of smaller-scale graphs to approximate large-scale data structures [5, 6], thereby providing effective models to improve data storage, memory access, and computational costs [7]. Three types of products are widely discussed, namely the Cartesian product, the Kronecker product, and the strong product [8]. Utilizing these product graphs, novel graph models can be designed to handle time-varying graph signals, facilitating the design of graph filters and signal prediction models [9–11].

Similar to general graph signal processing theory, frequency analysis is a fundamental issue in signal processing on product graphs, where the spectrum of factor graphs determines the frequency characteristics of the product graph. When factor graphs are unknown, designing specific models to learn these factor graphs becomes essential. Kadambari et al. developed product graph learning models constrained by data smoothness [12], as well as sparsity and rank [13]. With desired spectral templates, Einizade et al. proposed a product graph learning model that accommodates any type of product graph, even those with more than two factor graphs [14]. Once the factor graphs

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are obtained, frequency analysis on product graphs can be established. Cheng et al. introduced two SVD-based Fourier transform GFTs for the directed Cartesian product graph [15].

Compared to Cartesian product graphs, strong product graphs integrate data structures from the Kronecker product and thus offer a different frequency representation of graph signals [7]. They could integrate additional interactions between nodes in the graph and thus provide more precise models for time-varying signals [16]. However, as will be seen in our analysis, the eigenvalues of strong product graphs could be too clustered or repeated, performing as a frequency gathering problem. In applications of graph filters, similar eigenvalue gathering problem in general dense graphs could reduce the representation ability of the graph wavelet filter banks [17]. Additionally, it is proved that the number of distinguishable Laplacian eigenvalues plays a key role in determining the expression ability of the spectral graph neural network [18].

To solve the frequency gathering of strong product graphs, this paper introduces a novel concept of frequency equilibrium to measure the extent of frequency gathering in product graphs. Our theoretical analysis reveals that frequency gathering primarily originates from the Kronecker product. To mitigate this issue, we propose a new type of product called the weighted strong product. This graph is defined as a weighted combination of Cartesian and Kronecker products, which helps preserve geometric features while reducing the impact of frequency gathering caused by the Kronecker product.

In this paper, we present the following contributions:

- We present an analysis on the frequency distribution of strong products, and theoretically prove that the frequency gathering problem is highly likely to occur on product graphs.
- We introduce a novel concept of frequency equilibrium to evaluate the equilibrium of frequency distribution in product graphs.
- We propose a novel definition of the weighted strong product graph and establish a frequency equilibrium minimization model to determine the optimal weights for this product graph.
- Our experiments demonstrate that the proposed weighted strong product graphs, when using optimal weights, achieve better frequency equilibrium compared to the original strong product graphs.

The rest of this paper is organized as follows. Section 2 includes the background of the product graphs. In Section 3, we provide a theoretical analysis of product graph frequencies. We then introduce the concepts of frequency equilibrium and weighted strong product graph, and present an optimization model for calculating the optimal weights for the weighted strong product graph. Section 4 presents experimental results on random networks. Finally, Section 5 concludes this paper.

2. Preliminary

In this section, we briefly introduce the definitions of three fundamental types of product graphs: the Cartesian, Kronecker and strong product graphs. We also discuss the concept of graph frequency with respect to the adjacency matrix.

2.1 Adjacency Matrices of Product Graphs

Consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}|$ nodes, where $\mathcal{V}$ and $\mathcal{E}$ denote the sets of nodes and edges, respectively. The graph is stored by the weighted adjacency $\mathbf{A}=(a_{i,j})$ whose entry $a_{i,j}$ represents the weight assigned to the edge between node i and node j. A graph $\mathcal{G}$ is called unweighted, if $a_{i,j}=1$.

Consider two undirected graphs $\mathcal{G}_P = (\mathcal{V}_P, \mathcal{E}_P)$ and $\mathcal{G}_Q = (\mathcal{V}_Q, \mathcal{E}_Q)$ with $|\mathcal{V}_P| = P$ and $|\mathcal{V}_Q| = Q$ nodes, respectively. Let $\mathbf{A}_P \in \mathbb{R}^{P\times P}$ and $\mathbf{A}_Q \in \mathbb{R}^{Q\times Q}$ be the weighted adjacency matrix of $\mathcal{G}_P$ and $\mathcal{G}_Q$, respectively. The product of two graphs $\mathcal{G}_P$ and $\mathcal{G}_Q$ is denoted as $\mathcal{G}_\diamond = \mathcal{G}_P \diamond \mathcal{G}_Q$ with $|\mathcal{V}_P|\,|\mathcal{V}_Q| = PQ$ nodes and its adjacency is denoted as $\mathbf{A}_\diamond = \mathbf{A}_P \diamond \mathbf{A}_Q \in \mathbb{R}^{PQ\times PQ}$, where $\diamond$ is any kind of products.

For the Kronecker product, denoted as $\mathcal{G}_\times = \mathcal{G}_P \times \mathcal{G}_Q$, the adjacency matrix is

$$\mathbf{A}_\times = \mathbf{A}_P \otimes \mathbf{A}_Q, \tag{1}$$

where $\otimes$ denotes the Kronecker product of matrices.

We denote the Cartesian product of $\mathcal{G}_P$ and $\mathcal{G}_Q$ as $\mathcal{G}_\square = \mathcal{G}_P \square \mathcal{G}_Q$, the corresponding adjacency matrix is given by:

$$\mathbf{A}_\square = \mathbf{A}_P \otimes \mathbf{I}_Q + \mathbf{I}_P \otimes \mathbf{A}_Q. \tag{2}$$

For the strong product, denoted as $\mathcal{G}_\boxtimes = \mathcal{G}_P \boxtimes \mathcal{G}_Q$, we define its adjacency matrix as:

$$\begin{aligned}\mathbf{A}_\boxtimes &= \mathbf{A}_\square + \mathbf{A}_\times \\ &= \mathbf{A}_P \otimes \mathbf{I}_Q + \mathbf{I}_P \otimes \mathbf{A}_Q + \mathbf{A}_P \otimes \mathbf{A}_Q.\end{aligned} \tag{3}$$

2.2 Graph Frequency of Product Graphs

For an undirected graph $\mathcal{G}$ of order N, the adjacency matrix $\mathbf{A}$ is symmetric and its eigendecomposition is

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{\mathbf{T}}, \quad (4)$$

where $\mathbf{V} = [v_0, ..., v_{N-1}]$ is the orthogonal matrix of the N eigenvectors of $\mathbf{A}$, and the diagonal matrix $\mathbf{\Lambda} = \text{diag}\{\lambda_0, ..., \lambda_{N-1}\}$ consists of the corresponding eigenvalues. The eigenvalues λ_n of the adjacency $\mathbf{A}$ represent graph frequency, and the eigenvectors v_n represent the associated graph frequency components.

We can order the graph frequencies according to the total variation of the corresponding spectral component through

$$\mathbf{TV}_{\mathcal{G}}(v_k) = \left|1 - \frac{\lambda}{|\lambda_{\max}|}\right| \|v_k\|. \quad (5)$$

Using this, the graph frequencies are order as $\lambda_m < \lambda_n$, if the total variation of the associated spectral components are satisfy $\mathbf{TV}_{\mathcal{G}}(v_m) > \mathbf{TV}_{\mathcal{G}}(v_n)$ [19]. Therefore, if a graph has a real spectrum, the order of graph frequencies from low to high is $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{N-1}$.

Suppose that the eigendecomposition of the factor graph adjacency matrices $\mathbf{A}_P$ and $\mathbf{A}_Q$ are $\mathbf{A}_r = \mathbf{V}_r\mathbf{\Lambda}_r\mathbf{V}_r^T$, $r \in \{P, Q\}$, where $\mathbf{\Lambda}_P = \text{diag}\{\mu_0, \ldots, \mu_{P-1}\}$, $\mathbf{\Lambda}_Q = \text{diag}\{\eta_0, \ldots, \eta_{Q-1}\}$, and their diagonal entries are arranged in a nonincreasing order. If we denoted $\mathbf{V} = \mathbf{V}_P \otimes \mathbf{V}_Q$, then the eigendecomposition of the adjacency matrix $\mathbf{A}_{\Box}$ is [20]

$$\mathbf{A}_{\Box} = \mathbf{V}(\mathbf{\Lambda}_P \otimes \mathbf{I}_Q + \mathbf{I}_P \otimes \mathbf{\Lambda}_Q)\mathbf{V}^T, \quad (6)$$

the eigendecomposition of the adjacency matrix $\mathbf{A}_{\times}$ is

$$\mathbf{A}_{\times} = \mathbf{V}(\mathbf{\Lambda}_P \otimes \mathbf{\Lambda}_Q)\mathbf{V}^T, \quad (7)$$

and the eigendecomposition of the adjacency matrix $\mathbf{A}_{\boxtimes}$ is

$$\mathbf{A}_{\boxtimes} = \mathbf{V}(\mathbf{\Lambda}_P \otimes \mathbf{I}_Q + \mathbf{I}_P \otimes \mathbf{\Lambda}_Q + \mathbf{\Lambda}_P \otimes \mathbf{\Lambda}_Q)\mathbf{V}^T. \quad (8)$$

We can observe that the frequecny components of these three product graphs are consistent, but their corresponding frequencies are different. For the Cartesian, Kronecker and strong product graphs, their frequencies are $\mu_m + \eta_n$,$\mu_m\eta_n$ and $\mu_m + \eta_n + \mu_m\eta_n$, $0 \leq m \leq P-1$ and $0 \leq n \leq Q-1$, respectively. That is, the eigenvalues of the product graph are formulated by the eigenvalues of the factor graphs.

3 Weighted strong product graph

In this section, we theoretically analyze the frequency gathering problem in product graphs, propose the definition of frequency equilibrium, and introduce a novel weighted strong product graph and an optimal weighting model to alleviate the gathering problem.

3.1 Analysis on product graph frequencies

Consider the factor graph adjacency matrices $\mathbf{A}_P$ and $\mathbf{A}_Q$, with their eigenvalues μ_m, η_n, $0 \leq m \leq P-1$ and $0 \leq n \leq Q-1$, respectively. We denote $\mathbf{C}$=$(c_{m,n})$, $\mathbf{K}$=$(k_{m,n})$ and $\mathbf{S}$=$(s_{m,n})$, with $c_{m,n}$=μ_m+η_n, $k_{m,n}$=$\mu_m\eta_n$ and $s_{m,n}$=μ_m+η_n+$\mu_m\eta_n$. The (m, n)-th element in these eigenvalue matrices denotes the frequency associated with the (m, n)-th frequency component of the product graphs. The following propositions demonstrate that the strong product graph is highly likely to have repeated and gathered frequencies.

Proposition 3.1 For the strong product graph $\mathcal{G}_{\boxtimes}$, suppose its eigenvalues are $\mu_m + \eta_n + \mu_m\eta_n$, for $0 \leq m \leq P-1$, $0 \leq n \leq Q-1$. When $\mu_m = -1$ or $\eta_n = -1$, all eigenvalues of the m-th row or n-th column in $\mathbf{S}$ are -1.

Proof. For the eigenvalue μ_m+η_n+$\mu_m\eta_n$ of $\mathbf{S}$, if $\mu_m = -1$, we have μ_m+η_n+$\mu_m\eta_n$=-1+η_n−η_n=-1, $0 \leq n \leq Q-1$. That is, the eigenvalues of the m-th row in $\mathbf{S}$ are -1. The analysis for $\eta_n = -1$ is analogous to that for $\mu_m = -1$. ■

Proposition 3.2 For the Kronecker product graph $\mathcal{G}_{\times} = \mathcal{G}_P \times \mathcal{G}_Q$, suppose that the frequencies of the factor graphs are independent and let $\mathrm{m}_\mu := \max(\mu_m)$, $\mathrm{m}_\eta := \max(\eta_n)$. If the probabilities $P(|\mu_m| < c\mathrm{m}_\mu) = p_1$ and $P(|\eta_n| < c\mathrm{m}_\eta) = p_2$ are given, with $0 < c < 1$, we have $P(|\mu_m\eta_n| < c\mathrm{m}_\mu\mathrm{m}_\eta) \geq p_1 + p_2 - p_1p_2$.

Proof. Since the frequencies of the factor graphs are independent, considering $|\mu_m| < c\mathrm{m}_\mu$ or $|\eta_n| < c\mathrm{m}_\eta$, we have: $P(|\mu_m\eta_n| < c\mathrm{m}_\mu\mathrm{m}_\eta) \geq P(|\mu_m| < c\mathrm{m}_\mu)P(|\eta_n| < \mathrm{m}_\eta) + P(|\mu_m| < \mathrm{m}_\mu)P(|\eta_n| < c\mathrm{m}_\eta) - P(|\mu_m| < c\mathrm{m}_\mu)P(|\eta_n| < c\mathrm{m}_\eta) = p_1 + p_2 - p_1p_2$. ■

In fact, suppose that the frequencies of the factor graphs are uniformly distributed, i.e., for c=0.5, p_1=p_2=0.5. We have p_1+p_2−p_1p_2=0.75. Proposition 3.2 shows that the Kronecker product graph frequency could gather around the frequency band $|\mu_m\eta_n| < 0.5m_\mu\mathrm{m}_\eta$ with a probability greater than 0.75, even though the factor frequencies are uniformly distributed.

For the strong product graph $\mathcal{G}_{\boxtimes}$, its eigenvalues are $\mu_m + \eta_n + \mu_m\eta_n = (\mu_m+1)(\eta_n+1)-1$. Similar results to Proposition 3.2 for the strong product graphs could be obtained through a similar analysis, indicating similar frequency gathering problem of strong product graphs.

In the following, we propose a novel weighting scheme to alleviate the frequency gathering problem associated with strong product graphs.

3.2 Frequency equilibrium

To address the frequency gathering problem, a fundamental issue is how to measure its severity. In this section, we propose a novel definition of frequency equilibrium based on the difference between adjacent frequencies. If the frequencies are in equilibrium, then the difference between each pair of adjacent frequencies should be constant. Note that the row eigenvalue differences in the eigenvalue matrix of the product graph are primarily influenced by the eigenvalues of the second factor graph, while the column eigenvalue differences are predominantly influenced by the first factor graph. We then use the row and column range-to-scale ratios of the eigenvalues as reference frequency intervals. Frequency equilibrium is then defined by computing the distance of adjacent frequency differences from these row and column reference frequency intervals.

Definition 1 For the product graph $\mathcal{G}_\diamond = \mathcal{G}_P \diamond \mathcal{G}_Q$, its eigenvalue matrix is denoted by $\mathbf{Z}=(z_{m,n}) \in \mathbb{R}^{PQ\times PQ}$. We define the frequency equilibrium of $\mathcal{G}_\diamond$ in terms of $\mathbf{Z}$ by

$$\begin{aligned} \mathrm{F}_\diamond &= \sum_{m=1}^{P}\sum_{n'=1}^{Q-1}\left(|z_{m,n'+1}-z_{m,n'}|-d_{\diamond,1}\right)^2 \\ &+ \sum_{m'=1}^{P-1}\sum_{n=1}^{Q}\left(|z_{m'+1,n}-z_{m',n}|-d_{\diamond,2}\right)^2, \end{aligned} \tag{9}$$

where $d_{\diamond,1}=\frac{|\max(\mathbf{Z})-\min(\mathbf{Z})|}{P(Q-1)}$ and $d_{\diamond,2}=\frac{|\max(\mathbf{Z})-\min(\mathbf{Z})|}{(P-1)Q}$ are the column and row reference frequency interval of $\mathbf{Z}$, respectively.

The frequency equilibrium could be applied on different product matrices by letting $\mathbf{Z}$ be $\mathbf{C}=(c_{m,n})$, $\mathbf{K}=(k_{m,n})$, and $\mathbf{S}=(s_{m,n})$, respectively.

For convenience, we can rewrite (9) in matrix form:

$$\mathrm{F}_\diamond = \||\mathbf{Z}_3-\mathbf{Z}_1|-d_{\diamond,1}1\|_{\mathbf{F}}^2 + \||\mathbf{Z}_4-\mathbf{Z}_2|-d_{\diamond,2}1\|_{\mathbf{F}}^2 \tag{10}$$

where $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$ and $\mathbf{Z}_4$ are the sub-matrices of $\mathbf{Z}$ obtained by deleting the last column, the last row, the first column and the first row of $\mathbf{Z}$, respectively.

From Equations (9) and (10), it is clear that the closer the adjacent frequency differences are to the reference frequency intervals, the smaller the frequency equilibrium.

3.3 Weighted Strong Product Graph

Based on the analysis in the previous sections, we now introduce the concept of weighted strong product and determine the optimal weights to solve the frequency gathering problem.

Definition 2 Given two undirected factor graphs $\mathcal{G}_P = (\mathcal{V}_P, \mathcal{E}_P)$ and $\mathcal{G}_Q = (\mathcal{V}_Q, \mathcal{E}_Q)$ and two real numbers α and β. We define the weighted strong product of $\mathcal{G}_P$ and $\mathcal{G}_Q$ by $\mathcal{G}_\circledast = \mathcal{G}_P \circledast \mathcal{G}_Q$, and its adjacency matrix by:

$$\begin{aligned} \mathbf{A}_\circledast &= \alpha\mathbf{A}_\square + \beta\mathbf{A}_\times \\ &= \alpha(\mathbf{A}_P\otimes\mathbf{I}_Q + \mathbf{I}_P\otimes\mathbf{A}_Q) + \beta(\mathbf{A}_P\otimes\mathbf{A}_Q). \end{aligned} \tag{11}$$

Theorem 3.3 Suppose that the eigendecomposition of the matrices $\mathbf{A}_P$ and $\mathbf{A}_Q$ are $\mathbf{A}_r=\mathbf{V}_r\mathbf{\Lambda}_r\mathbf{V}_r^T$, $r \in \{P,Q\}$, where $\mathbf{\Lambda}_P=\mathrm{diag}\{\mu_0,\ldots,\mu_{P-1}\}$, $\mathbf{\Lambda}_Q=\mathrm{diag}\{\eta_0,\ldots,\eta_{Q-1}\}$. If we denoted $\mathbf{V}=\mathbf{V}_P\otimes\mathbf{V}_Q$, then the characteristics(frequency components) of $\mathcal{G}_\circledast$ are the columns of $\mathbf{V}$. The frequencies of $\mathcal{G}_\circledast$ are then given by $\alpha\mathbf{C}+\beta\mathbf{K}$.

Proof. Since $\mathbf{V}_P$ and $\mathbf{V}_Q$ are orthogonal matrices, we have $\mathbf{V}_P\mathbf{V}_P^T=\mathbf{E}$, $\mathbf{V}_Q\mathbf{V}_Q^T=\mathbf{E}$. And by using the properties of Kronecker product: $(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D})=\mathbf{AC}\otimes\mathbf{BD}$ and $\mathbf{A}^T\otimes\mathbf{B}^T=(\mathbf{A}\otimes\mathbf{B})^T$, we have:

$$\begin{aligned} \mathbf{A}_\circledast &= \alpha(\mathbf{A}_P\otimes\mathbf{I}_Q + \mathbf{I}_P\otimes\mathbf{A}_Q) + \beta(\mathbf{A}_P\otimes\mathbf{A}_Q) \\ &= \alpha(\mathbf{V}_P\mathbf{\Lambda}_P\mathbf{V}_P^T)\otimes(\mathbf{V}_Q\mathbf{I}_Q\mathbf{V}_Q^T) \\ &+ \alpha(\mathbf{V}_P\mathbf{I}_P\mathbf{V}_P^T)\otimes(\mathbf{V}_Q\mathbf{\Lambda}_Q\mathbf{V}_Q^T) \\ &+ \beta(\mathbf{V}_P\mathbf{\Lambda}_P\mathbf{V}_P^T)\otimes(\mathbf{V}_Q\mathbf{\Lambda}_Q\mathbf{V}_Q^T) \\ &= \alpha\mathbf{V}\mathbf{C}\mathbf{V}^T + \beta\mathbf{V}\mathbf{K}\mathbf{V}^T \\ &= \mathbf{V}[\alpha\mathbf{C}+\beta\mathbf{K}]\mathbf{V}^T. \end{aligned} \tag{12}$$

where $\mathbf{C}=\mathbf{\Lambda}_P\otimes\mathbf{I}_Q+\mathbf{I}_P\otimes\mathbf{\Lambda}_Q$, $\mathbf{K}=\mathbf{\Lambda}_P\otimes\mathbf{\Lambda}_Q$. ■

From Theorem 3.3 and (12), it is clear that the graph frequency components of the weighted strong product graph $\mathcal{G}_\circledast$ is the same as that of the three product graphs.

Finally, we consider how to determine the weights α and β of the weighted strong product with minimal frequency equilibrium according to its eigenvalue matrix

$(\alpha\mathbf{C}+\beta\mathbf{K})\in\mathbb{R}^{P\times Q}$. Letting $\mathbf{Z}=\alpha\mathbf{C}+\beta\mathbf{K}$ in Definition 1, we then solve the following weighting minimization model.

$$\begin{aligned}\min_{\alpha,\beta}\ &\|\alpha|\mathbf{C}_3-\mathbf{C}_1|-d_1\mathbb{1}\|_{\mathbf{F}}^2+\|\alpha|\mathbf{C}_4-\mathbf{C}_2|-d_2\mathbb{1}\|_{\mathbf{F}}^2\\ &+\|\beta|\mathbf{K}_3-\mathbf{K}_1|-d_1\mathbb{1}\|_{\mathbf{F}}^2+\|\beta|\mathbf{K}_4-\mathbf{K}_2|-d_2\mathbb{1}\|_{\mathbf{F}}^2,\end{aligned} \tag{13}$$

where $d_1=\frac{|\max(\mathbf{C})+\max(\mathbf{K})-\min(\mathbf{C})-\min(\mathbf{K})|}{P(Q-1)}$ and $d_2=\frac{|\max(\mathbf{C})+\max(\mathbf{K})-\min(\mathbf{C})-\min(\mathbf{K})|}{(P-1)Q}$, and the matrices $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ and $\mathbf{C}_4$ ($\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{K}_3$ and $\mathbf{K}_4$) are obtained by deleting the last column, the last row, the first column and the first row of $\mathbf{C}$ ($\mathbf{K}$), respectively.

Problem (13) is an unconstrained quadratic problem and can thus be solved directly using standard optimization methods. It is important to note that the optimal weighting parameters α and β may be negative, which could result in negative edge weights in the adjacency matrix. In cases where edge weights are required to be positive, our weighting model can be adjusted accordingly. We will focus on developing constrained models for real-world applications in our future work.

4 Experiment

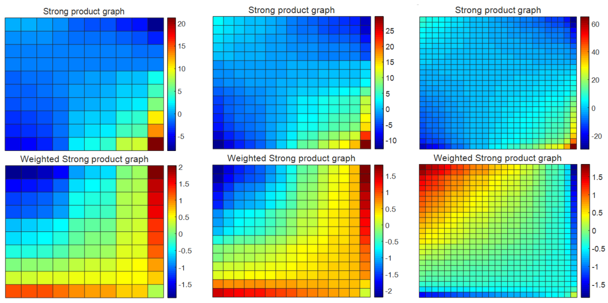
In this section, we present experimental results to show the frequency equilibrium of the weighted strong product graphs, and compare with the original product graphs.

Consider the product of two factor graphs $\mathcal{G}_P$ and $\mathcal{G}_Q$ generated by unweighted Erdös-Rényi random graphs. We calculate the frequency equilibrium $\mathbf{F}_{\square}, \mathbf{F}_{\times}$ and $\mathbf{F}_{\boxtimes}$ of the Cartesian, Kronecker and strong product graphs, respectively.

TABLE 1. Frequency equilibrium of product graphs

	$\mathbf{F}_{\square}$	$\mathbf{F}_{\times}$	$\mathbf{F}_{\boxtimes}$	$\mathbf{F}_{\circledast}$
$P=10, Q=10$	134.799	367.765	506.394	19.215
$P=15, Q=15$	200.446	791.359	989.384	18.535
$P=25, Q=25$	923.951	6246.816	7162.900	19.148

The sparsity of random graphs is determined by a parameter $p\in[0,1]$. In our experiments, we generate a dense factor graph $\mathcal{G}_P$ with $p=0.8$ and a sparse factor graph $\mathcal{G}_Q$ with $p=0.3$. We then calculate the graph frequency equilibrium of the product of $\mathcal{G}_P$ and $\mathcal{G}_Q$ with varying numbers of nodes. As can be seen in Table 1, the proposed weighted strong product exhibits the smallest frequency equilibrium compared to other products. Additionally, our weighting model results in more closely distributed eigenvalues, stabilizing the frequency equilibrium of the weighted strong products around 20. In contrast, the frequency equilibrium of other products increases sharply with more nodes. Although the frequency equilibrium of Cartesian product graphs is smaller than that of Kronecker and strong products, their larger eigenvalue magnitudes still make it significantly higher than that of the proposed weighted product.



(a) (b) (c)

FIGURE 1. The eigenvalue matrix color-maps of the strong and weighted strong product of random factor graphs. (a) $P=10, Q=10$**. (b)** $P=15, Q=15$**. (c)** $P=25, Q=25$**.**

To visually illustrate the variation in the frequency gathering phenomenon due to differing nodes, Figure 1 presents the eigenvalue matrix color-maps of $\mathbf{S}$ and $\alpha\mathbf{C}+\beta\mathbf{K}$ of the strong and weighted strong products listed in Table 1. As can be seen in the first row of Figure 1, the color map of each strong product eigen-matrix predominantly features blue, indicating that the eigenvalues cluster around the high frequency band (colored in blue). In contrast, the eigenvalue matrix color-maps of the weighted strong product graphs, as displayed in the second row of Figure 1, shows a more uniform distribution of colors, with no dominant color observed. This demonstrates that the proposed weighted strong product and weighting model effectively alleviate the frequency gathering problem.

5 Conclusion

In this paper, we provide a theoretical analysis of the frequency gathering problem in product graphs. To address this issue, we introduce the concept of frequency equilibrium to quantify the severity of frequency gathering. We then present the definition of the weighted strong product and formulate a weighting minimization model based on frequency equilibrium to determine the optimal weights for this product. Experimental results demonstrate that the proposed weighted strong product graphs achieves better and more stable frequency equilibrium compared to other product graphs.

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