FAST SWING-UP CONTROL OF INERTIA WHEEL PENDULUM SYSTEMS BASED ON MOTION PLANNING

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Abstract:

This paper presents a fast swing-up control method for inertia wheel pendulum systems based on motion planning. Firstly, perform a state transformation using the system property of differential flatness, and transform the motion planning problem between the initial point and the target point in the state space into a curve interpolation of the differential flatness output with relevant boundary conditions satisfied. Then, a polynomial interpolation approach is adopted to derive the optimization problem for minimum-time motion planning, which is solved based on the bisection method and particle swarm optimization algorithm to obtain the minimum-time motion trajectory. On that basis, the system's dynamic characteristics and non-minimum phase property are thoroughly analyzed to design an effective trajectory tracking controller for tracking the reference optimal trajectory, thus the fast swing-up control of the inertia wheel pendulum system can be achieved. Finally, the proposed method is verified with simulation test and compared with a direct swing-up controller without motion planning. The test results demonstrate the feasibility and superiority of the proposed method, and the swing-up time under related system constraints can be decreased significantly.

Keywords:

Inertia wheel pendulum; Fast swing-up; Differential flatness; Bisection method; Motion planning; Tracking control

1. Introduction

The inertia wheel pendulum system has two degrees of freedom, i.e., the swing of the pendulum and the rotation of the inertia wheel, but only one control input, namely, the driving torque of the inertia wheel, and the swing of the pendulum is driven by the rotation of the inertia wheel, indirectly [1]. As for a typical underactuated system [2], the inertia wheel pendulum can not only be used a benchmark to research various nonlinear control theories [3], but also has rich engineering application prospects in several fields, such as ground transportation, aerospace, and industrial automation[4]. Consequently, inertia wheel pendulum systems have attracted extensive attention from both academia and industry.

Belascuen et al. [5] studied design methods of the inertia wheel pendulum system, and explored the optimization of its mechanical structure under given drive motors and inertia wheel diameter to increase the system's recovery angle. Subsequently, Su et al. [6] derived the dynamic model of the inertia wheel pendulum using the Lagrangian modeling method and proposed a finite difference discretization model for the identification of system parameters, as well as some simulation results to verify the effectiveness of the proposed approach. Furthermore, Sureshkumar et al. [7] achieved a reconfigurable inertia wheel pendulum device by adjusting the pendulum's center of mass and analyzed stability conditions under various center-of-mass parameter configurations. Additionally, Meri et al. [8] addressed the estimation accuracy issue for the pendulum's angular velocity affected by the measurement noise, they compared the function of a reduced-order estimator and a differentiator in system stabilization, and the results show that the reduced-order state estimator performs better in both transient and steady-state conditions. The aforementioned studies primarily focus on aspects such as system structure design, dynamic modeling, stability analysis, and state estimation, but control methods for the inertia wheel pendulum was not explored in-depth, particularly the critical issue of effective swing-up control still requires the corresponding solution methods.

To fixed this problem, Srinivas et al. [9] proposed two practical swing-up control methods, where the first method treated the pendulum's oscillation as a perturbation from the bottom equilibrium point, and the second one is based on the interconnection and damping assignment to achieve passivity control, and both methods exhibit faster response characteristics compared to energy-based control schemes. Afterwards, Sowman et al. [10] designed a high-performance nonlinear model predictive controller for the swing-up control of the inertia wheel pendulum system, and derived an explicit version to solve the problem of the real-time computational limitations, thus the precise approximation for the required control inputs is realized and the effective compromise between real-time performance and accuracy can be carried out. Furthermore, Montoya et al. [11] investigated classical control methods for the inertia wheel pendulum system, such as PID control and state feedback control, with the stability of their proposed controllers being analyzed based on Lyapunov theory, and control laws were designed to ensure global asymptotic stability of the closed-loop system, thus effective control of the pendulum from arbitrary initial positions to the desired upright position can be achieved. Although these methods exhibit good performance in swing-up control of the inertia wheel pendulum system, improving swing-up control efficiency remains an open problem.

In view of this, a motion planning-based swing-up control method for the inertia wheel pendulum system is proposed in this study. By fully considering the system's motion constraints, the minimum-time motion trajectory planning is performed, and a tracking controller for the planned trajectory is designed based on the analysis of the system's non-minimum phase characteristics, thus the swing-up control efficiency can be improved. The remainder of the paper is organized with the following structure. Section 2 describes the swing-up control problem of the inertia wheel pendulum system, and the minimum-time motion planning is given in Section 3. Afterwards, Section 4 designs the corresponding trajectory tracking controller to realize the fast swing-up control of the pendulum. Finally, the feasibility and superiority of the proposed method are validated in Section 5 through simulation test, and valuable conclusions are provided in Section 6.

2. Problem Description

To analyze the motion of the inertia wheel pendulum system, its dynamic model is firstly established. As shown in Fig.1, the inertia wheel pendulum system primarily consists of a pendulum arm and an inertia wheel, and the torque generated by the rotation of the inertia wheel acts on the pendulum arm, which drives the swinging motion of the pendulum. Therefore, using the Euler-Lagrange modeling approach, the system's dynamic equations can be derived as follows

$$m_{11}\ddot{\theta}_1 + m_{12}\ddot{\theta}_2 + c(\theta_1) = 0 \tag{1}$$

$$m_{21}\ddot{\theta}_1 + m_{22}\ddot{\theta}_2 = \tau \tag{2}$$

where $m_{11} = m_1 l_1^2 + m_2 l_2^2 + I_1 + I_2$, $m_{12} = m_{21} = m_{22} = I_2$, $c(\theta) = -(m_1 l_1 + m_2 l_2) g \sin\theta_1$, m_1 and m_2 denote the masses of the pendulum arm and the inertia wheel, respectively, I_1 and I_2 denote the moments of inertia for the pendulum arm and the inertia wheel, respectively, l_1 and l_2 represent the distances from the mass centers of the pendulum arm and the inertia wheel to the pivot point, respectively, τ represents the output torque of the reaction wheel's drive motor, θ_1 is the angle of the pendulum arm, and θ_2 is the rotation angle of the inertia wheel.

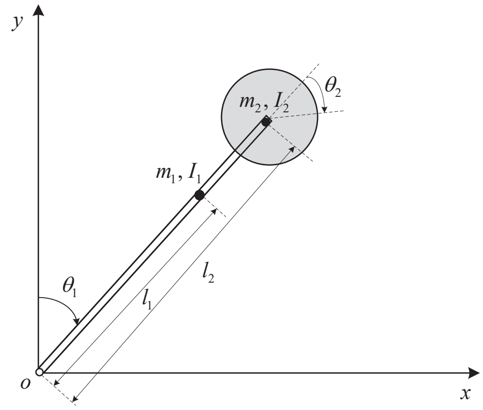

FIGURE 1. Schematic diagram of the inertia wheel pendulum system

Assume that the pendulum angle is $\theta_1(t_0)$ and the inertia wheel angle is $\theta_2(t_0)$ at the initial time t_0, respectively. For clarity and without loss of generality, let the initial moment when the system swing be the time zero, thus the initial values of the pendulum angle θ_1 and the inertia wheel angle θ_2 satisfy

$$\theta_1(t_0) = \theta_1(0) = \theta_1^0 \tag{3}$$

$$\theta_2(t_0) = \theta_2(0) = 0 \tag{4}$$

where θ_1^0 represents the initial value of the pendulum angle.

To achieve the swing-up of the system successfully, the pendulum angle should arrive the upright position after time t_f, and the final values of the pendulum angle and the inertia wheel angle satisfy.

$$\theta_1(t_\mathrm{f}) = 0, \theta_2(t_\mathrm{f}) = \theta_2^\mathrm{f} \tag{5}$$

where θ_2^f represents the inertia wheel angle after the system's swing-up. Since the specific value of this variable does not affect the swing-up process of the system, no specialized control is required.

Moreover, to ensure a smoother start and finish of the swing-up process, the angular velocities of both the pendulum and the inertia wheel are set to zero at the initial and final moments, thus we have

$$\dot{\theta}_1(0) = \dot{\theta}_1(t_f) = 0 \tag{6}$$

$$\dot{\theta}_2(0) = \dot{\theta}_2(t_f) = 0 \tag{7}$$

Additionally, taking actuator constraints into account, the angular velocities of the pendulum and the inertia wheel should satisfy the following limits

$$|\dot{\theta}_1(t)| \leq \dot{\theta}_1^{\max}, |\dot{\theta}_2(t)| \leq \dot{\theta}_2^{\max} \tag{8}$$

where $\dot{\theta}_1^{\max}$ and $\dot{\theta}_2^{\max}$ represent the maximum allowable angular velocities for the pendulum swing and the inertia wheel rotation, respectively.

In summary, the problem of fast swing-up for the inertia wheel pendulum system can be formulated as follows: for a given $\theta_1^0 \in (0, \theta_{1\max}^0)$, find the control input τ that minimizes the motion time required for the system to move from the initial state (3) and (4) to the upright state (5) with the system motion constraints (6)-(8) being satisfied, thereby realize the fast swing-up motion.

3. Minimum-time Motion Planning

From the system dynamics equations (1) and (2), it can be concluded that the inertia wheel pendulum system exhibits typical underactuated characteristics, which lacks of a direct control input corresponding to the pendulum angle, and its motion planning must rely on the nonlinear coupling relationship between the pendulum swing and the inertia wheel rotation. To this end, this study uses the differential flatness property of the system to perform state transformation, and converts the planning of inertia wheel pendulum state variables into a problem of planning for flat outputs to be solved.

3.1. State Transformation

Equation (1) can be further expressed in the following state-space equation form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})u \tag{9}$$

where $\boldsymbol{x} = [\theta_1\ \dot{\theta}_1\ \theta_2\ \dot{\theta}_2]^{\mathrm{T}}$, $\boldsymbol{f}(\boldsymbol{x}) = [\dot{\theta}_1\ J_1 \sin\theta_1\ \dot{\theta}_2\ 0]^{\mathrm{T}}$, $\boldsymbol{g}(\boldsymbol{x}) = [0\ J_2\ 0\ 1]^{\mathrm{T}}$, $J_1 = (m_1 l_1 + m_2 l_2)g/m_{11}$, $J_2 = -I_2/m_{11}$, and $u = \ddot{\theta}_2$.

Analysis reveals that the aforementioned system is a differential flat system with the following equation [12] as its flat output

$$y = h(\boldsymbol{x}) = \theta_1 - J_2\theta_2 \tag{10}$$

Based on the above analysis, let $\boldsymbol{\xi} = [\xi_1\ \xi_2\ \xi_3\ \xi_4]^{\mathrm{T}}$, and the following coordinate transformation being adopted

$$\boldsymbol{\xi}_i = \phi_i(\boldsymbol{x}) = L_f^{i-1}h(\boldsymbol{x}), i = 1, 2, 3, 4 \tag{11}$$

The system (11) can be further transformed into the following form

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \dot{\xi}_3 = \xi_4 \\ \dot{\xi}_4 = p(\boldsymbol{\xi}) + q(\boldsymbol{\xi})u \end{cases} \tag{12}$$

where $\xi_1 = \theta_1 - J_2\theta_2$, $\xi_2 = \dot{\theta}_1 - J_2\dot{\theta}_2$, $\xi_3 = J_1\sin\theta_1$, $\xi_4 = J_1\dot{\theta}_1\cos\theta_1$, $q(\boldsymbol{\xi}) = J_1 J_2\cos\theta_1$, and $p(\boldsymbol{\xi}) = -J_1\dot{\theta}_1^2\sin\theta_1 + J_1^2\sin\theta_1\cos\theta_1$.

The mapping relationship between the differential flat output states and the original system states is

$$\theta_1 = \arcsin\left(\frac{\xi_3}{J_1}\right) \tag{13}$$

$$\dot{\theta}_1 = \frac{\xi_4}{J_1\sqrt{1 - (\xi_3/J_1)^2}} \tag{14}$$

$$\theta_2 = \frac{1}{J_2}\left[\arcsin(\xi_3/J_1) - \xi_1\right] \tag{15}$$

$$\dot{\theta}_2 = \frac{1}{J_2}\left[\frac{\xi_4}{J_1\sqrt{1 - (\xi_3/J_1)^2}} - \xi_2\right] \tag{16}$$

We focus on studying the motion of the inertia wheel pendulum system in the upper half-plane, and assumes that the pendulum angle variation range satisfies $\theta_1 \in (-\pi/2, \pi/2)$, thus $0 \leq |\xi_3| \leq J_1$ holds true. Consequently, $1 - (\xi_3/J_1)^2 > 0$ is valid, and the aforementioned transformation remains singularity-free within the scope of this study.

To sum up, through this transformation, the motion planning problem of the inertia wheel pendulum system between the initial state-space point $\boldsymbol{x}(0)$ and the target point $\boldsymbol{x}(t_f)$ can be converted into a curve interpolation problem in the differential flat output space that satisfies the relevant boundary conditions.

3.2. Trajectory Parameterization

Polynomial function is applied as the differential flat output, thus the specific expression can be given as follows

$$y = \sum_{i=1}^{s} a_{i-1}t^{i-1} \tag{17}$$

where $t \in [0, t_f]$, $a_i (i = 1, 2, ..., s)$ are constants representing the undetermined coefficients of the polynomial, and s is a given constant related to the degree of the polynomial, which satisfies $s > 8$.

Based on the initial and final states of the system, the following redundant undetermined coefficient matrix equation can be constructed

$$A\boldsymbol{a} = \begin{bmatrix} \boldsymbol{\xi}(0) \\ \boldsymbol{\xi}(t_\mathrm{f}) \end{bmatrix} \tag{18}$$

where $\boldsymbol{a} = [a_0 \ a_1 \ ... \ a_{s-1}]^\mathrm{T}$, and $A \in R^{8\times s}$ satisfies

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 2 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & 6 & 0 & ... & 0 \\ 1 & t_\mathrm{f} & t_\mathrm{f}^2 & t_\mathrm{f}^3 & t_\mathrm{f}^4 & ... & t_\mathrm{f}^{s-1} \\ 0 & 1 & 2t_\mathrm{f} & 3t_\mathrm{f}^2 & 4t_\mathrm{f}^3 & ... & (s-1)t_\mathrm{f}^{s-2} \\ 0 & 0 & 2 & 6t_\mathrm{f} & 12t_\mathrm{f}^2 & ... & \dfrac{(s-1)!}{(s-3)!}t_\mathrm{f}^{s-3} \\ 0 & 0 & 0 & 6 & 24t_\mathrm{f} & ... & \dfrac{(s-1)!}{(s-4)!}t_\mathrm{f}^{s-4} \end{bmatrix}$$

Meanwhile, based on the relevant state constraints (3)-(5) at the initial and terminal moments of the inertia wheel pendulum, as well as the relationship between the system states and the flat outputs, the following initial and terminal states of the flat outputs can be obtained

$$\boldsymbol{\xi}(0) = \begin{bmatrix} \theta_1^0 & 0 & J_1 \sin\theta_1^0 & 0 \end{bmatrix}^\mathrm{T} \tag{19}$$

$$\boldsymbol{\xi}(t_\mathrm{f}) = \begin{bmatrix} -J_2\theta_2^\mathrm{f} & 0 & 0 & 0 \end{bmatrix}^\mathrm{T} \tag{20}$$

According to the pendulum angular velocity constraint (8) and the relationship between the system states and the flat outputs (14) and (16), the following inequality constraints can be obtained

$$\left| \frac{\dddot{y}}{J_1\sqrt{1-(\ddot{y}/J_1)^2}} \right| \le \dot{\theta}_1^{\max} \tag{21}$$

$$\left| \frac{\dddot{y}}{J_1 J_2\sqrt{1-(\ddot{y}/J_1)^2}} - \frac{\dot{y}}{J_2} \right| \le \dot{\theta}_2^{\max} \tag{22}$$

Therefore, the minimum-time swing-up motion planning for the inertia wheel pendulum system can be transformed into the following optimization problem.

$$\begin{aligned} &\min \ t_\mathrm{f}(a_0, a_1, ..., a_{s-1}) \\ &\text{s.t.} \begin{cases} A\boldsymbol{a} = [\boldsymbol{\xi}(0), \boldsymbol{\xi}(t_\mathrm{f})]^\mathrm{T} \\ \left| \dfrac{\dddot{y}}{J_1\sqrt{1-(\ddot{y}/J_1)^2}} \right| \le \dot{\theta}_1^{\max} \\ \left| \dfrac{\dddot{y}}{J_1 J_2\sqrt{1-(\ddot{y}/J_1)^2}} - \dfrac{\dot{y}}{J_2} \right| \le \dot{\theta}_2^{\max} \end{cases} \end{aligned} \tag{23}$$

3.3. Trajectory optimization solution

It can be seen from (23) that the objective function of this optimization problem is quasi-convex, while the relevant constraints are convex, which can be regarded as a standard quasi-convex optimization problem [13], and the optimal solution can be obtained by solving a series of convex feasibility problems. Meanwhile, considering the multidimensional nonlinear characteristics of the problem, the convex feasibility problems can be iteratively addressed using the particle swarm optimization algorithm.

Additionally, a larger s leads to more variables to be optimized, which can also increase the complexity of the solution. Conversely, a smaller s reduces the number of variables to be optimized, which may result in the non-existence of an optimal solution. In this study, balancing the necessary conditions for the existence of an optimal solution and the requirement to reduce computational complexity, $s = 10$ is chosen such that the variables to be optimized are 10 parameters, namely, a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, and a_9. Note that the number of optimization variables can be further reduced based on the equality constraints in the system.

To this end, by substituting (17) and its derivatives into the equality constraints in (23), we obtain

$$a_0 = \theta_1^0 \tag{24}$$

$$a_1 = 0 \tag{25}$$

$$a_2 = \frac{J_1}{2}\sin\theta_1^0 \tag{26}$$

$$a_3 = 0 \tag{27}$$

$$\begin{aligned} &a_0 + a_1 t_\mathrm{f} + a_2 t_\mathrm{f}^2 + a_3 t_\mathrm{f}^3 + a_4 t_\mathrm{f}^4 + a_5 t_\mathrm{f}^5 + a_6 t_\mathrm{f}^6 + \\ &a_7 t_\mathrm{f}^7 + a_8 t_\mathrm{f}^8 + a_9 t_\mathrm{f}^9 = -J_2\theta_2^\mathrm{f} \end{aligned} \tag{28}$$

$$\begin{aligned} &4a_4 t_\mathrm{f}^3 + 5a_5 t_\mathrm{f}^5 + 6a_6 t_\mathrm{f}^5 + 7a_7 t_\mathrm{f}^6 + 8a_8 t_\mathrm{f}^7 + 9a_9 t_\mathrm{f}^8 = \\ &-J_1 t_\mathrm{f}\sin\theta_1^0 \end{aligned} \tag{29}$$

$$5a_5 t_\mathrm{f}^4 + 12a_6 t_\mathrm{f}^5 + 21a_7 t_\mathrm{f}^6 + 32a_8 t_\mathrm{f}^7 + 45a_9 t_\mathrm{f}^8 = 2J_1 t_\mathrm{f}\sin\theta_1^0 \tag{30}$$

$$2a_6t_f^5 + 7a_7t_f^6 + 16a_8t_f^7 + 30a_9t_f^8 = -J_1t_f\sin\theta_1^0 \quad (31)$$

Clearly, the four parameters a_0 , a_1, a_2 and a_3 can be determined by (24), (25), (26) and (27), respectively. Since θ_2^f can take any arbitrary value, thus (28) has no practical effect. The remaining six parameters are subject to the three constraints given by (29)-(31). Once a_7 , a_8 and a_9 are determined, a_6 can be derived from (31), followed by determining a_5 from (30) and a_4 from (29). This process allows all trajectory coefficients to be fully determined.

On that basis, the following steps can be employed to solve the optimization problem

(1) Given the allowable error p_c for the bisection method's termination, along with the upper bound t_u and lower bound t_l for t_f, where t_u is determined empirically and t_l can be derived from the following equation

$$t_l = \theta_1^0/\dot{\theta}_1^{\max} \quad (32)$$

(2) Calculate the swing-up motion time of the system as follows

$$t_f = (t_l + t_u)/2 \quad (33)$$

(3) Taking a_7, a_8 and a_9 as the unknown variables to be optimized (where a_0, a_1, a_2, a_3, a_4, a_5 and a_6 can be obtained from (24)-(27) and (29)-(31)), and minimizing the violation of the constraints in (23) as the objective function, a particle swarm optimization algorithm with N particles is iterated M times to obtain the constraint violation measure e_c .

(4) If $e_c < e_c^{\max}$, it indicates that there exists a motion trajectory satisfying the current t_f and the relevant constraints. In this case, assign the current t_f to the upper bound t_u of the bisection method. Otherwise, it means no feasible motion trajectory satisfies the current t_f and constraints, so assign the current t_f to the lower bound t_l of the bisection method, where $e_c^{\max}$ is a given small constant.

(5) Calculate the error e_b between the upper and lower bounds of the bisection method as follows:

$$e_b = t_u - t_l \quad (34)$$

If $e_b > p_c$, return to step (2) to continue solving; otherwise, the process terminates, and the current t_f and the optimal trajectory parameters a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 and a_9 are output, thereby generating the minimum-time swing-up trajectory for the inertia wheel pendulum system.

4. Design of the Stable Tracking Controller

After obtaining the time-optimal swing-up trajectory, it is necessary to design a tracking controller to achieve fast swing-up control of the system. It can be seen from the system's dynamic equations (1) and (2), the inertia wheel pendulum system has two degrees of freedom (θ_1 and θ_2) but only one control input (the inertial wheel's driving torque τ), which makes it a typical underactuated system, thus the pendulum angle can only be indirectly regulated through the control of the inertial wheel.

To fixed this problem, two PD tracking controllers for both the pendulum and the inertia wheel are designed, separately, then combines their outputs as the final control input, as follows

$$\tau = k_1(\theta_1-\theta_1^d)+k_2(\dot{\theta}_1-\dot{\theta}_1^d)+k_3(\theta_2-\theta_2^d)+k_4(\dot{\theta}_2-\dot{\theta}_2^d) \quad (35)$$

where θ_1^d and $\dot{\theta}_1^d$ represent the desired pendulum angle and angular velocity, respectively, which are determined by the planned time-optimal trajectory, θ_2^d and $\dot{\theta}_2^d$ denote the desired inertia wheel angle and angular velocity, respectively, which are also derived from the planned time-optimal trajectory, and k_1, k_2, k_3 and k_4 are four controller design constants, the sign assignments of which will be analyzed in detail below.

For convenience, the principle of pendulum motion control is analyzed using the state near the upright equilibrium point of the system as an example. As shown in Fig.2(a), when the pendulum reaches the upright position, i.e., the pendulum angle θ_1 equals the desired value, the inertial wheel's control torque τ should be zero. When the pendulum deviates to the left of the upright position, as shown in Fig.2(b), i.e., when the pendulum angle θ_1 is less than the desired value, the inertia wheel's control torque τ should be negative, which generates a torque that causes the pendulum to move clockwise, thereby increasing the pendulum angle. When the pendulum deviates to the right of the upright position, as shown in Fig.2(c), i.e., when the pendulum angle θ_1 exceeds the desired value, the inertial wheel's control torque τ should be positive, which produces a counterclockwise torque to reduce the pendulum angle. Therefore, k_1 should take a positive value. Similarly, k_2 should also be positive. For the motion control of the inertia wheel, due to the non-minimum phase characteristics of the system, conventional negative feedback control would prevent the pendulum's motion of swinging up. Thus, a positive feedback control strategy should be adopted. Specifically, when the inertia wheel angle θ_2 is smaller than the desired value θ_2^d, the control torque should be negative; when θ_2 exceeds θ_2^d, the control torque should be positive, thus k_3 and k_4 should also be assigned positive values.

5. Simulation Results

To verify the effectiveness of the proposed fast swing-up control method based on motion planning for the inertia wheel

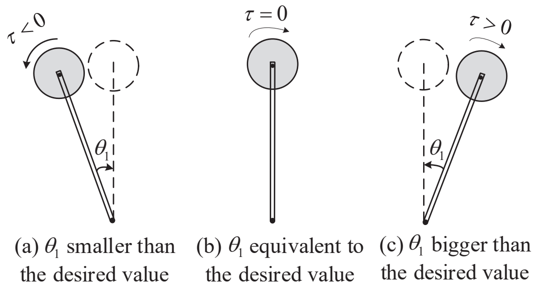

FIGURE 2. Control principle of the pendulum

pendulum system, simulation test were conducted using Matlab/Simulink. The results were compared with a swing-up controller without motion planning to demonstrate the advantages of the proposed approach. The structural parameters of the system were selected as shown in Tab.1, with the initial pendulum angle being set to $\theta_1^0 = \pi/6$ and the velocity constraints for the pendulum and inertia wheel being set to $\dot{\theta}_1^{\max} = 12$ and $\dot{\theta}_2^{\max} = 2000$, respectively. In addition, the key parameters in particle swarm optimization algorithm are set with $N = 2000$ and $M = 100$, and the tracking controller parameters are set with $k_1 = 80$, $k_2 = 20$, $k_3 = 0.0001$, and $k_4 = 0.0001$. The simulation results are presented in Fig.3-Fig.5.

TABLE 1. Structure parameters of the system

Parameters	Value	Units
l_1	0.063	m
l_2	0.125	m
m_1	0.02	kg
m_2	0.3	kg
I_1	0.000047	$\mathrm{kg\cdot m^2}$
I_2	0.000032	$\mathrm{kg\cdot m^2}$

As indicated by the red dashed line in Fig.3, the planned time-optimal trajectory enables the pendulum angle θ_1 to rapidly move from its initial value to zero in just 0.168s. Furthermore, the designed tracking controller effectively ensures that the actual pendulum angle closely follows the planned trajectory, as the black solid line show in Fig.3. In contrast, a direct PD controller without motion planning requires more than 1s to drive the pendulum angle θ_1 from the initia value to zero, as depicted by the blue dash-dotted line in Fig.3.

Additionally, as shown in Fig.4 and Fig.5, the pendulum angular velocity (red dashed line in Fig.4) determined by the

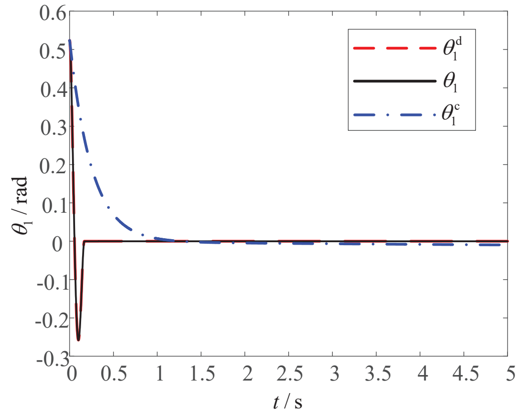

FIGURE 3. Time response of θ_1

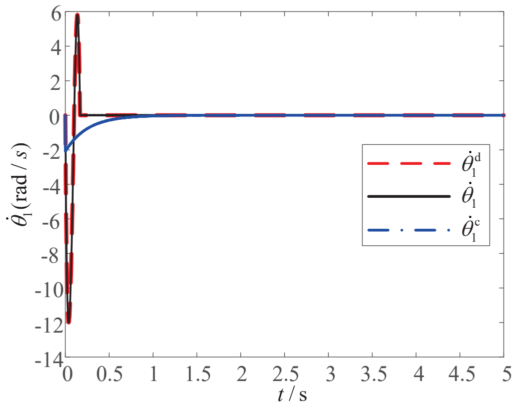

FIGURE 4. Time response of $\dot{\theta}_1$

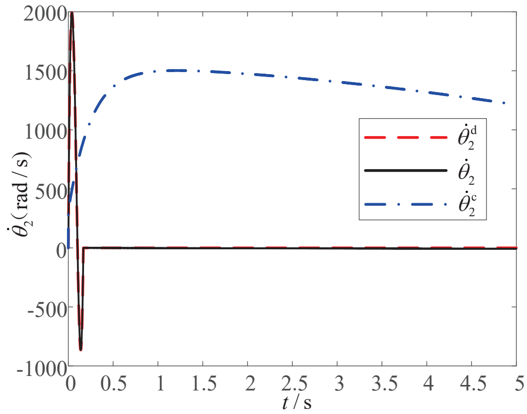

FIGURE 5. Time response of $\dot{\theta}_2$

planned time-optimal trajectory satisfies the $|\dot{\theta}_1| \leq \dot{\theta}_1^{\max}$ constraint, while the inertia wheel angular velocity (red dashed line in Fig.5) meets the $|\dot{\theta}_2| \leq \dot{\theta}_2^{\max}$ constraint. Under the designed tracking controller, the actual system trajectories effectively track the desired reference trajectories.

By contrast, when using a direct PD controller without motion planning, although the state trajectories still comply with the constraints, they remain significantly distant from the constraint limits, which indicates a failure to fully exploit the system's motion potential. Therefore, the proposed motion planning-based fast swing-up control method can achieve minimum-time swing-up to the upright position while satisfy system state constraints, demonstrating the feasibility and superiority of the proposed approach.

6. Conclusions

A fast swing-up control method for inertia wheel pendulum systems based on motion planning is proposed in this study. By employing differential flatness theory and considering system motion constraints, an optimization model for minimum-time swing-up trajectory generation is derived, which is solved by a bisection method combined with particle swarm optimization. On that basis, an effective trajectory tracking controller is designed through dynamic analysis to achieve fast swing-up of the inertia wheel pendulum. Simulation results demonstrate that the proposed method enables fast swing-up while satisfies system constraints, and significantly reduces the swing-up time compared to controllers without motion planning.

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References

[1] M. W. Spong, P. Corke, and R. Lozano, "Nonlinear control of the reaction wheel pendulum", Automatica, Vol 37, pp. 1845-1851, 2001.

[2] L. Navarro, and M. Maggiore, "Hybrid stabilization of closed orbits for a class of underactuated mechanical systems", IEEE Trans. Autom. Control, Vol 69, No. 10, pp. 6864-6879, 2024.

[3] D. Gutierrez-Oribio, A. Mercado-Uribe and J. A. Moreno, "Reaction wheel pendulum control using fourth-order discontinuous integral algorithm", Int. J. Robust Nonlinear Control, Vol 31, No. 1, pp. 185-206, 2021.

[4] M. Hofer, M. Muehlebach and R. D'Andrea, "The one-wheel cubli: a 3D inverted pendulum that can balance with a single reaction wheel", Mechatronics, Vol 91, pp. 102965, 2023.

[5] G. Belascuen, and N. Aguilar, "Design, modeling and control of a reaction wheel balanced inverted pendulum", Proceeding of IEEE Biennial Congr. Argentina, pp. 1-9, 2018.

[6] C. Y. Su, C. C. Peng, A. A. Ravankar and A. Ravankar, "Dynamics modelling and parameter identification of a reaction wheel based pendulum", Proceeding of Int. Conf. Robot. Comput., pp. 271-274, 2020.

[7] B. M. Meri, C. G. Vicentin, G. Rickli, M. Klement, J. G. Ferreira, L. V. Bellinaso and G. S. Azzolin, "Comparison of angular velocity estimation methods for reaction wheel inverted pendulum control", Proceeding of Seminar Power Electron. Control, pp. 1-6, 2024.

[8] V. Sureshkumar, K. Harib, A. Oseni and A. Aladawi, "Design and analysis of an inverted pendulum driven by a reaction wheel", Proceeding of IEEE/ASME Int. Conf. Adv. Intell. Mechatronics, pp. 440-446, 2024.

[9] K. N. Srinivas, and L. Behera, "Swing-up control strategies for a reaction wheel pendulum", Int. J. Syst. Sci., Vol 39, No. 12, pp. 1165-1177, 2008.

[10] O. D. Montoya, and W. Gil-Gonzalez, "Nonlinear analysis and control of a reaction wheel pendulum: Lyapunov-based approach", Eng. Sci. Technol. Int. J., Vol 23, No. 1, pp. 21-29, 2020.

[11] D. Ruiz, D. Vazquez, R. Marquez and M. Bernal, "Nonlinear control of inertia wheel pendulum via exact feedback linearization", Proceeding of Indian Control Conf., pp. 30-35, 2024.

[12] G. He, X. Liang, and T. Su, Dynamics and control of underactuated robots, Posts and Telecom Press, Beijing, China, 2020.

[13] X. B. Zhang, Y. C. Fang, and N. Sun, "Minimum-time trajectory planning for underactuated overhead crane systems with state and control constraints", IEEE Trans. Ind. Electron., Vol 61, No. 12, pp. 6915-6925, 2014.